NUMERICAL ANALYSIS OF SIMPLIFIED RELIC-BIREFRINGENT GRAVITATIONAL WAVES

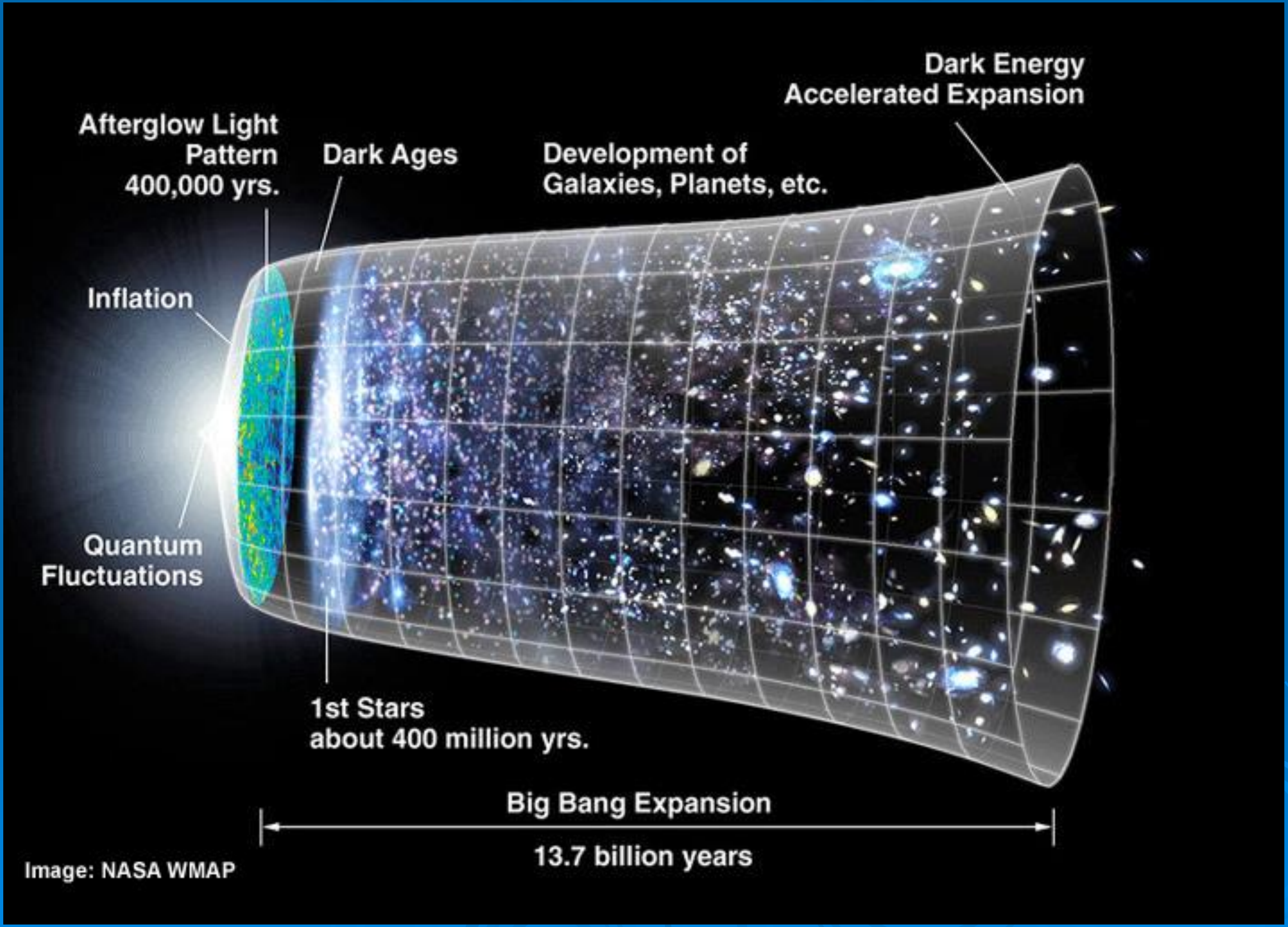
Rafael de la Torre
Dr. David Garrison

3/29/10

In The Beginning ….

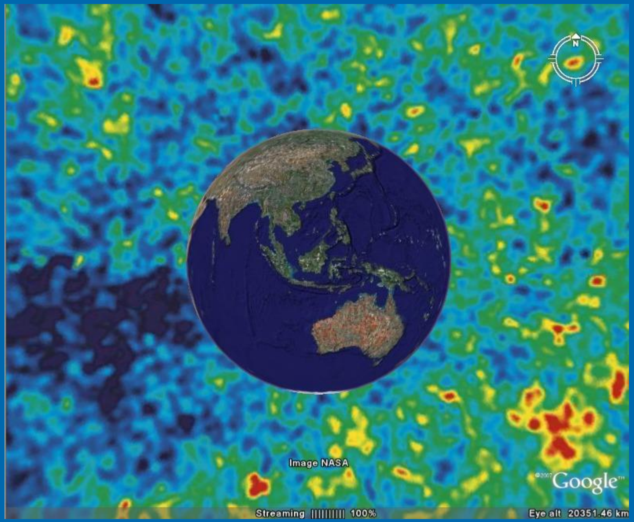
- Standard model of the universe (Big Bang)
  - Originally discovered by Edwin Hubble's observations of distant galaxies
    - Red-shifted in all directions = accelerating away
  - Decades of supporting evidence
    - Confirmation of Hubble's observations from various sources
    - Cosmic Microwave Background (CMB) observations

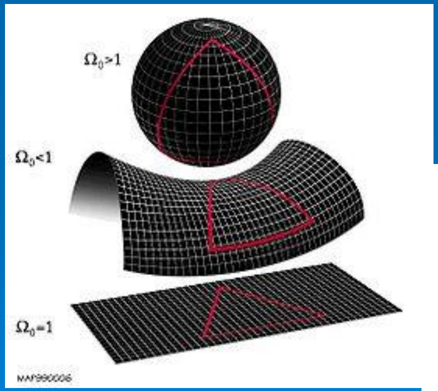
Standard Model

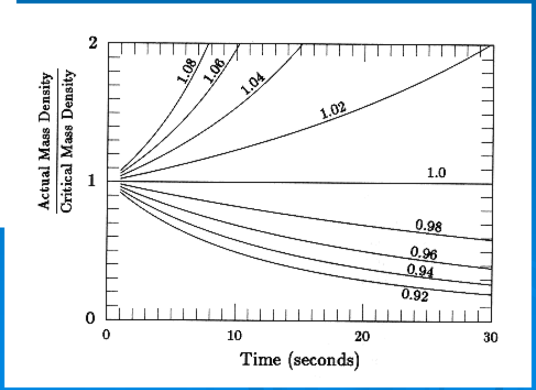


Big “Problems”

- Horizon and Smoothness Problems: The Universe exhibits large-scale homogeneity that shouldn’t exist





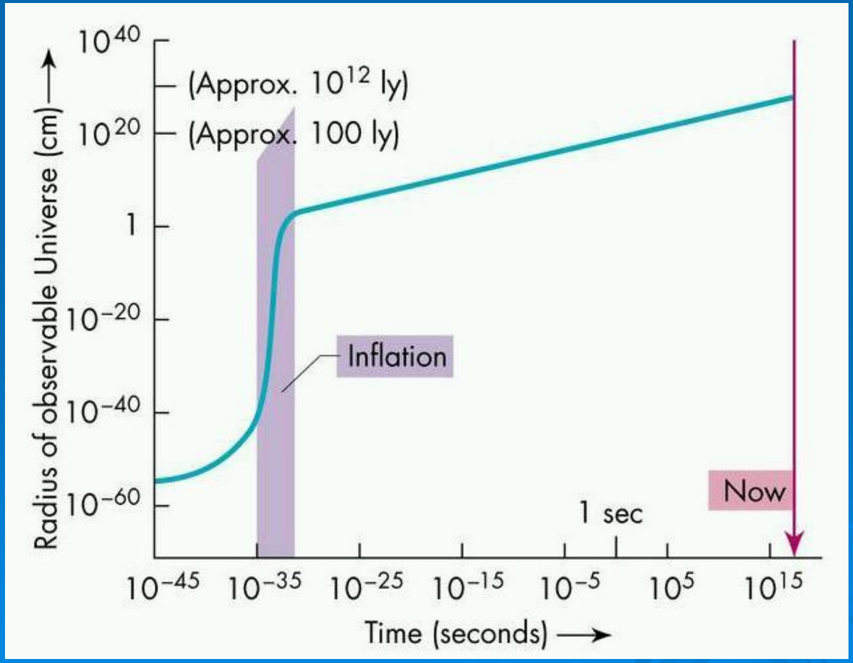


- Flatness Problem: The Universe shouldn’t be ~flat today

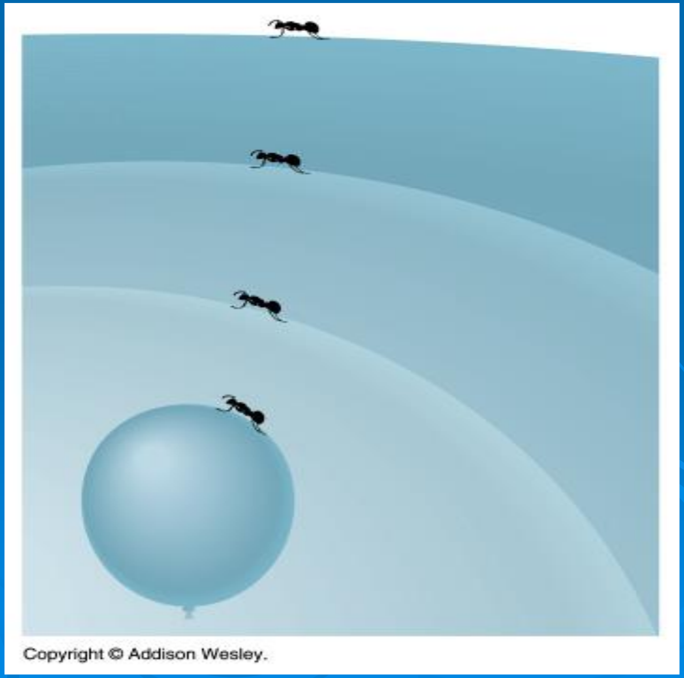
Inflation Theory

- Solves problems with standard model
  - Proposed by MIT Physicist Alan Guth in 1980
- Extremely rapid expansion of the universe in a very short time span (~10^{-35} to ~10^{-32} s)

Horizon and Smoothness Problems

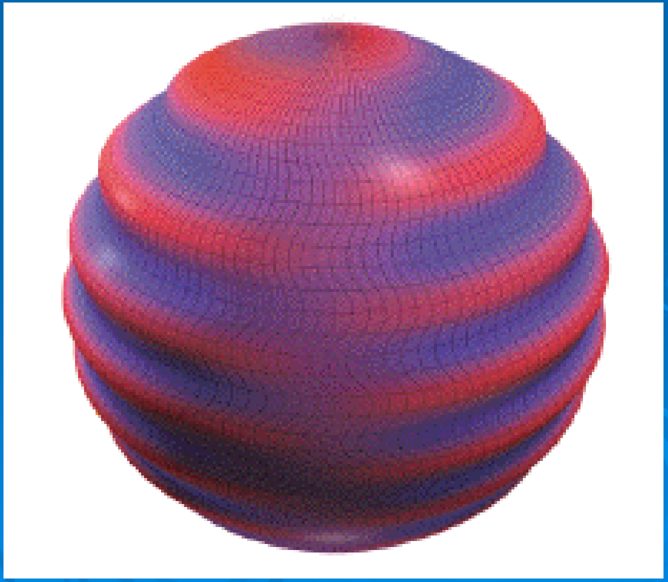


Flatness Problem



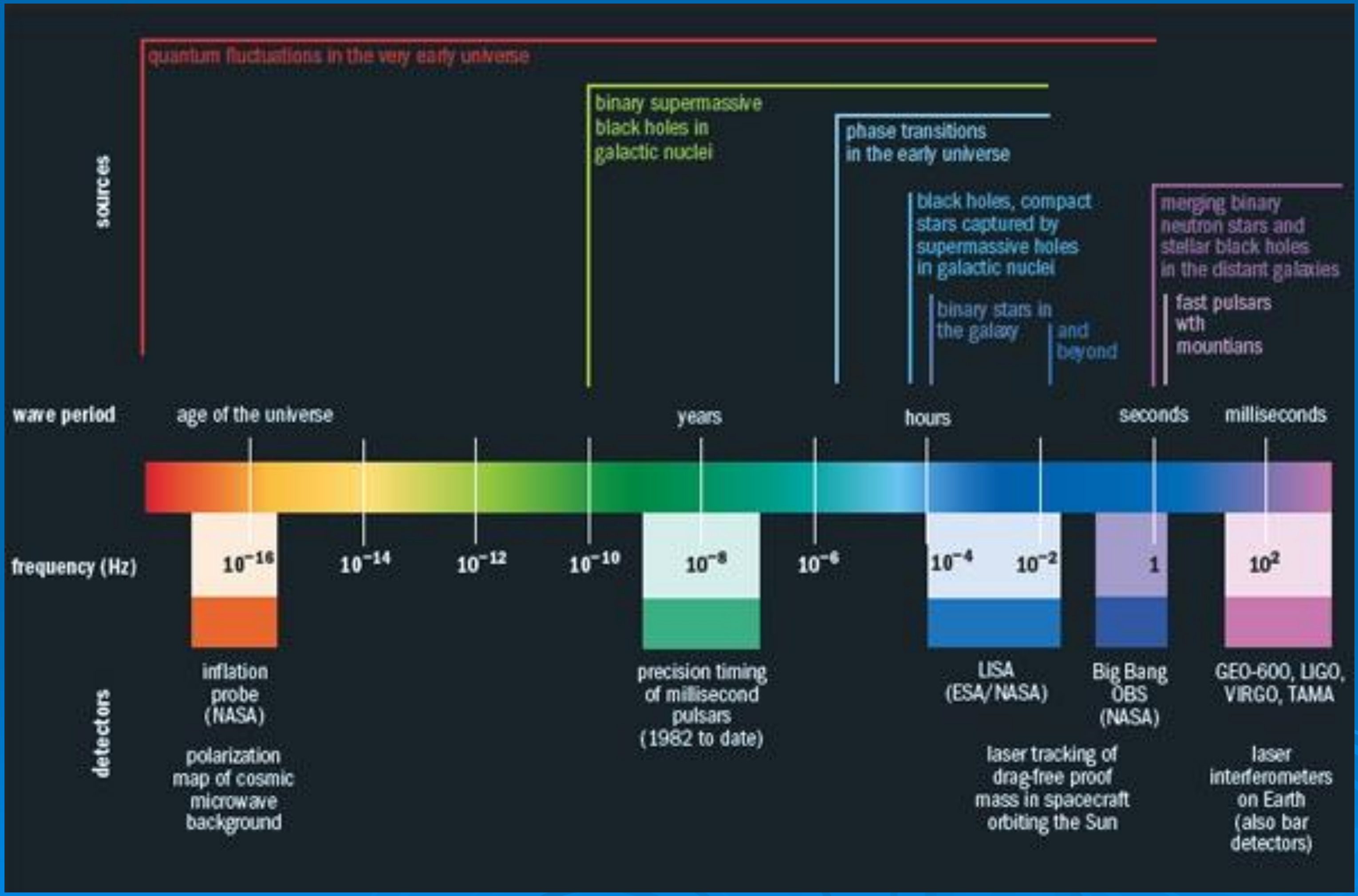
Gravitational Waves

- Prediction of Einstein's General Relativity
  - Space-Time curved by mass
    - Larger mass corresponds to greater curvature
  - Einstein's equations can be written as wave equations using adequate assumptions
- Very large mass-energy objects in motion can produce "ripples" in space-time
  - Ripples = Gravity Waves (GWs)
  - Possible Sources Include:
    - Black holes, massive/dense stars
    - **Very early universe**



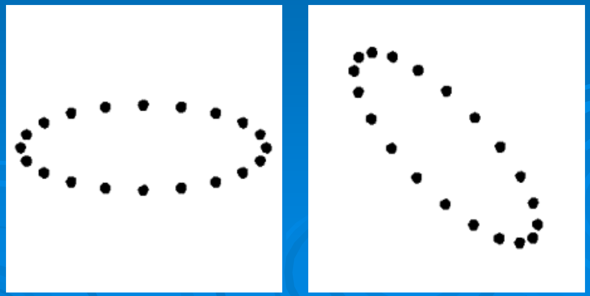
Relic GWs are remnants of the exponential inflation of the very early universe

Gravitational Spectrum



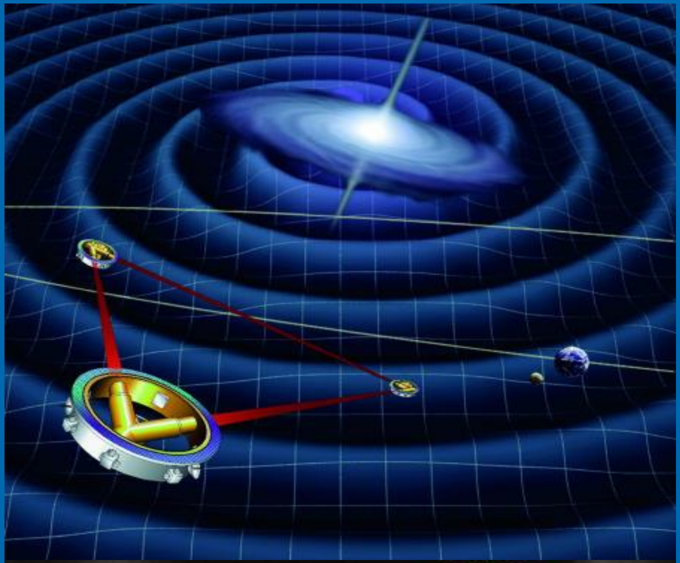
Relic GWs

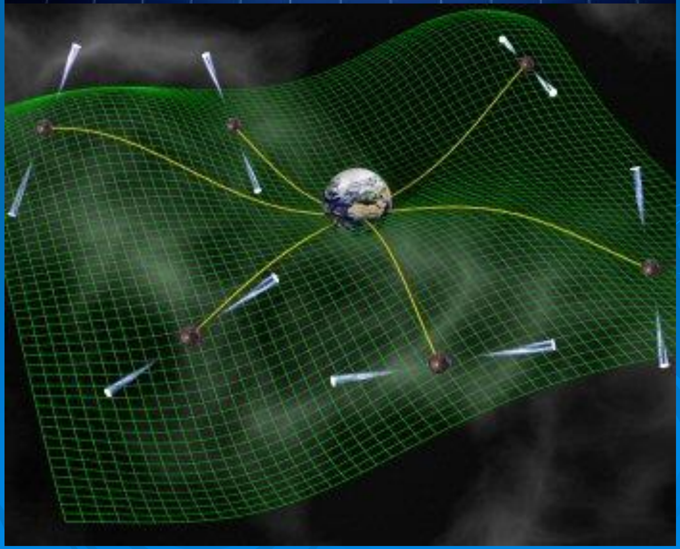
- Possibly associated with cosmological structure formation
  - Leptogenesis
  - GRMHD turbulence
- Recent work by Alexander, et al. adds potential Quantum Gravitation effects
  - Inclusion of QG terms predicts very small anisotropies in space-time during early universe
  - Anisotropies produce *Birefringent* (BRF) GWs with different polarities



Detection

- Relic GW theory suggests a CMB analog may exist for gravitational radiation
  - Cosmic Gravitational Wave Background (CGWB)
- Detecting relic-BRF GWs would provide evidence of inflation and QG effects
- GW spectrum is needed to focus detectors on certain amplitude vs. frequency regions





Relic GW Spectrum

- Relic GWs result in tensor perturbations to the ST metric

$$ds^2 = -(1+2\varphi)dt^2 + \omega_i dx^i dt + a(t)\left[\left((1+2\psi)\delta_{ij} + h_{ij}\right)dx^i dx^j\right]$$

$$ds^2 = a^2(\eta)[d\eta^2 - (\delta_{ij} + h_{ij})dx^i dx^j]$$

- Spectrum determined using QM approach
  - Tensor perturbations become operators in Quantum-Gravitational expectation function

$$\langle 0|h_{ij}(\eta,x)h^{ij}(\eta,x)|0\rangle = \left[\frac{C}{\pi}\right]^2 \frac{1}{2}\int_0^\infty k^2 \sum_{p=L,R} \left|h_k^p(\eta)\right|^2 \frac{dk}{k}$$

Mean-square amplitude of spectrum →

$$h^2(k,\eta) = \left[\frac{4l_{PL}}{\sqrt{\pi}}k\right]^2 \frac{1}{2}\sum_{p=L,R}\left|h_k^p(\eta)\right|^2$$

← ***Mode Function is primary variable to determine!***

- Computation of spectrum is only required at some conformal time (η) during inflation

Mode Functions

- Composed of GW equation and "effective" scale factor

$$h_p(\eta) = \frac{g_p(\eta)}{z_p(\eta)}$$

  - Scale factor and effective scale factor have the following form during inflation

$$a(\eta) = l_o |\eta|^{1+\beta} \quad \text{for} \quad -\infty \le \eta \le \eta_{iz}; -2 \le \beta \le -1$$

$$z_p = a(\eta)\sqrt{1 - \lambda_p \theta k \eta}$$

- GW equation is key to determine mode function!
  - Given by equation for 1-D Harmonic oscillator

$$\frac{d^2}{d\eta^2} g_p + \left(k^2 - Veff_p\right) g_p = 0$$

Standard and Simplified BRF Relic GWs

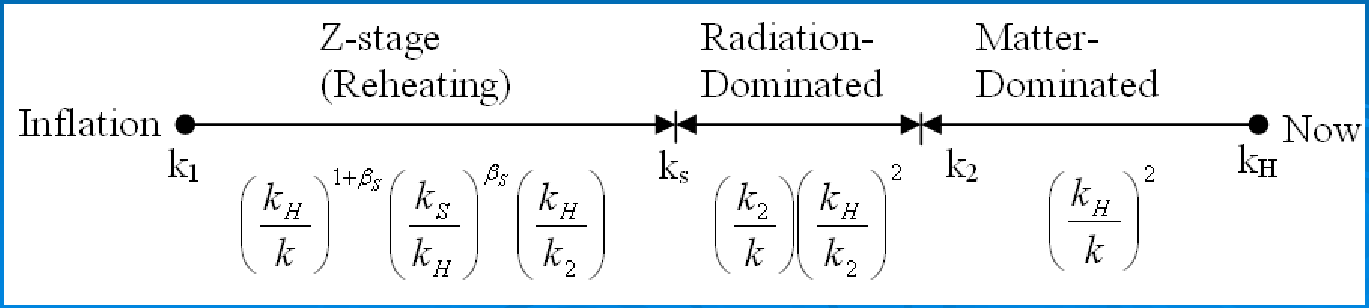
	Standard GW	Simplified BRF GW - $k\theta\eta << 1$
Veff	$$\frac{a''(\eta)}{a(\eta)} = \frac{\beta(1+\beta)}{\eta^2}$$	$$\frac{z''(\eta)_p}{z(\eta)_p} = \frac{\beta(1+\beta)}{\eta^2} + \lambda_p \frac{2(1+\beta)}{\eta} \frac{k\theta}{(1-\lambda_p 2k\theta\eta)} + \frac{(k\theta)^2}{(1-\lambda_p 2k\theta\eta)^2}$$ Simplified $$\frac{z''(\eta)_p}{z(\eta)_p} = \frac{\beta(1+\beta)}{\eta^2} + \lambda_p \frac{2(1+\beta)k\theta}{\eta} + (k\theta)^2$$
General Analytical Solution	$$g(\eta) = C_1\sqrt{k\eta}e^{-ik\eta}J_{(\beta+\frac{1}{2})}(k\eta) + C_2\sqrt{k\eta}e^{+ik\eta}Y_{(\beta+\frac{1}{2})}(k\eta)$$	$$g_p(\eta) = C_1 M_{\kappa_p,\mu}\left(\beta+1-\kappa_p, 2\beta+2, 2ik\eta\sqrt{1-\theta^2}\right) + C_2 W_{\kappa_p,\mu}\left(\beta+1-\kappa_p, 2\beta+2, 2ik\eta\sqrt{1-\theta^2}\right)$$
Behavior	For $k^2 << Veff \quad g(\eta) \to Ca(\eta)$, For $k^2 >> Veff \quad g(\eta) \to Ce^{-ik\eta}$	

Where, $b = 2^{2+\beta}\left|1+\beta\right|^{-(1+\beta)}$; Where, $\kappa_p = \lambda_p \left|(1+\beta)\theta / \sqrt{\theta^2 - 1}\right|$

Red-shift of RMS Amplitude

- Spectral amplitude at inflation is red-shifted for sub-Hubble radius wave lengths to any time since
  - Method developed by Grischuk for standard GW spectrum
- Red-shift is function of scale factors at different "breakpoints"
  - Breakpoints correspond to transition between different eras
  - Red-shift during each era determined by general relation

$$1 + z = \frac{\nu_e}{\nu_o} = \frac{k_e}{k_o} = \frac{a_{now}(\eta)}{a_{then}(\eta)}$$



Variable Properties

- Frequency & wave number - 1e-20 $\leq \nu \leq$ 1e+10
  - Low-frequency limit corresponds to wave lengths > current Hubble radius
  - High-frequency limit established from relationship

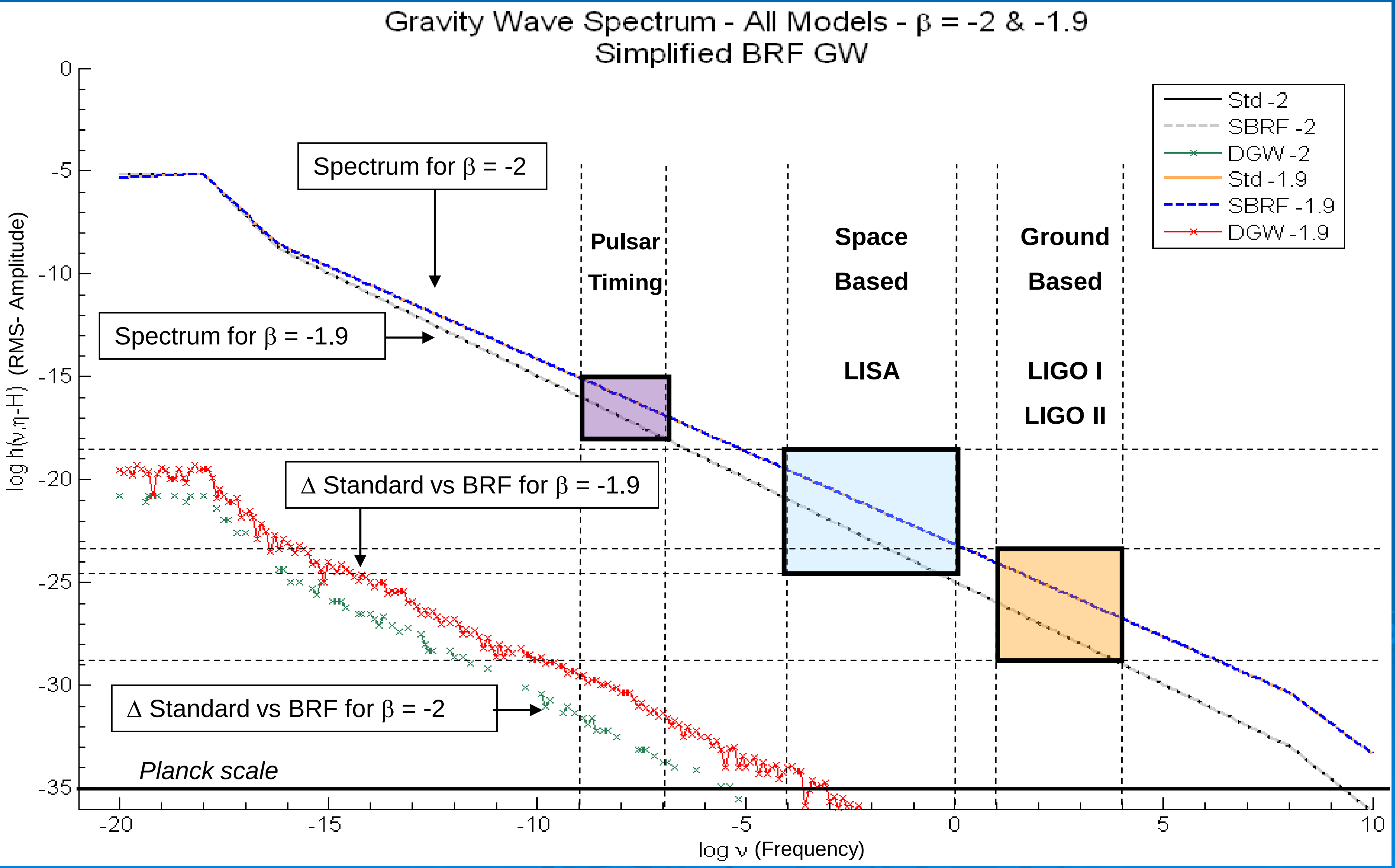
$$b\frac{l_{Pl}}{l_o}\left(\frac{\nu}{\nu_H}\right)^{2+\beta} < 1$$

- Power-law expansion parameter - β = -2 and -1.9
  - Theoretical restriction - 1+ β < 0
  - Values chosen encompass expected range
    - -2 Corresponds to de Sitter Universe (No Matter, + Λ)
    - -1.9 incorporates slow-roll approximation (falls within this range)
- BRF Parameter - $\theta \leq$ 1e-05
  - Composed of QG and string theory parameters
  - Values > than this result in non-linear Veff
- Conformal Time - $|\eta|$ < 1e-18
  - Due to numerical instabilities in high frequency regime
  - Results from constraint k η θ << 1

Numerical approach

- Used Numerical routines available in MATLAB
  - Applied built-in Bessel & Gamma functions
  - Imported and modified special complex Kummer & Gamma function modules from user community
- Evolved particular solutions of GW equations using variable constraints
  - Calculated mode function and spectrum at initial conformal time for each step through GW equations
  - Red-shifted initial spectrum using Grishchuk's method depending on values of ν
  - Plotted resulting spectrum and differences between models
- Validity check
  - BRF GW algorithms numerically computed with theta = 0 to compare against standard values
  - Compared results to theoretical approximations where possible

Relic GW Spectrum
RMS Amplitude vs Frequency



Conclusions

- GW spectrum for *simplified* BRF GWs does not appear to be significantly different than the standard GW spectrum at the present time
  - Indicates that BRF GW ~ Standard GW for $k|\eta|\theta << 1$ during instant of GW creation
  - If BFR GW were created, they may not be distinguishable unless θ and $k|\eta|\theta$ are different than assumptions
- Review of general BRF analytical solutions imply limit on conformal time of GW creation (at least for RH wave)
  - $k|\eta|\theta < ½$ so $|\eta| < 5e\text{-}15$

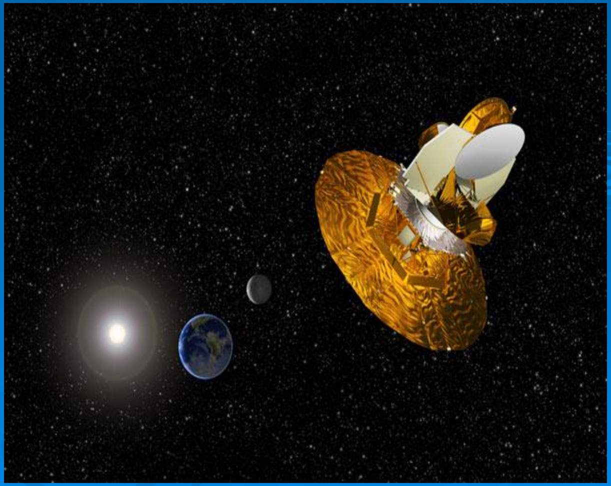
Forward Work

- Apply red-shift for accelerating Universe as proposed by Zhang, et al. for completeness
- Examine numerical solution to general BRF GW ODE
- Variable θ
  - Examine non-linear behavior and implications for $\theta > 1e{-}05$
- Determine Spectrum at 3 minutes to support GRMHD work
  - Target red-shift for this period
  - θ may have more implications here
- Numerical algorithms
  - Code and Transport to other platforms
    - C++/Cactus for ongoing GRMHD work
  - Investigate alternatives if necessary

Final Thoughts

- Between 1946 – 1964, various papers predicting CMB were written.
  - Background temperature estimated from below 20 K & 45 K
  - One paper predicted it was detectable
- In 1964 Arno Penzias and Robert Wilson measure unexpected radio noise while testing a microwave receiver
  - Published paper on noise
  - Later identified by Robert Dicke *as CMB*
  - Helped refine Big Bang model (3 K)
- NASA began development of the COBE satellite in 1976 and launched it in 1989

When and where will the Bell Labs moment happen for the CGWB?



References

- Contained in Published Paper

Numerical analysis of simplified relic-birefringent gravitational waves
David Garrison and Rafael de la Torre 2007 *Class. Quantum Grav.* **24** 5889