Numerical Cosmology: Building a Dynamical Universe

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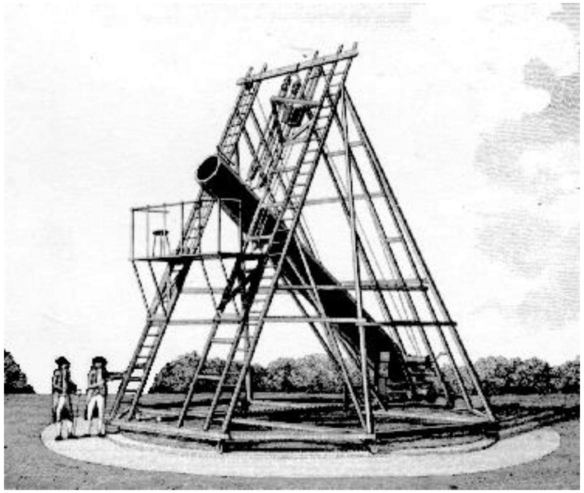
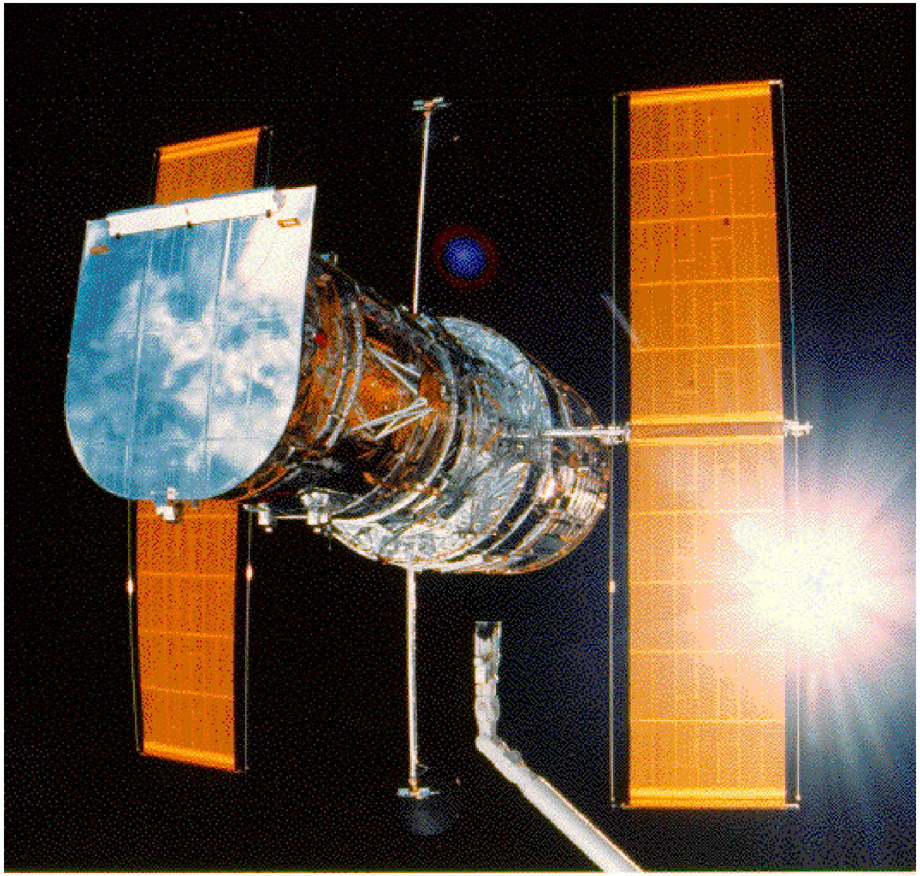
The Beginning

Where the Heck did all that come from?

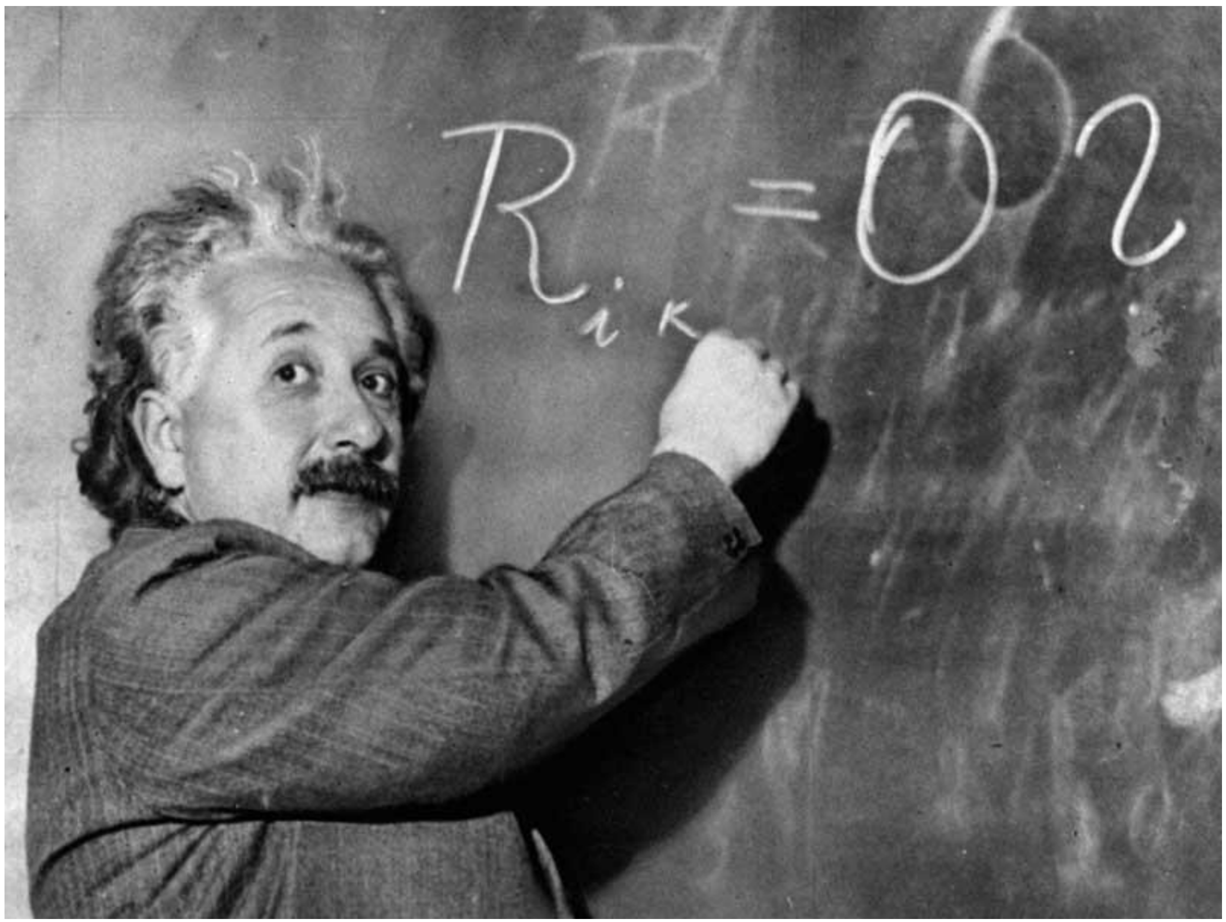
First Observatories

New Technologies

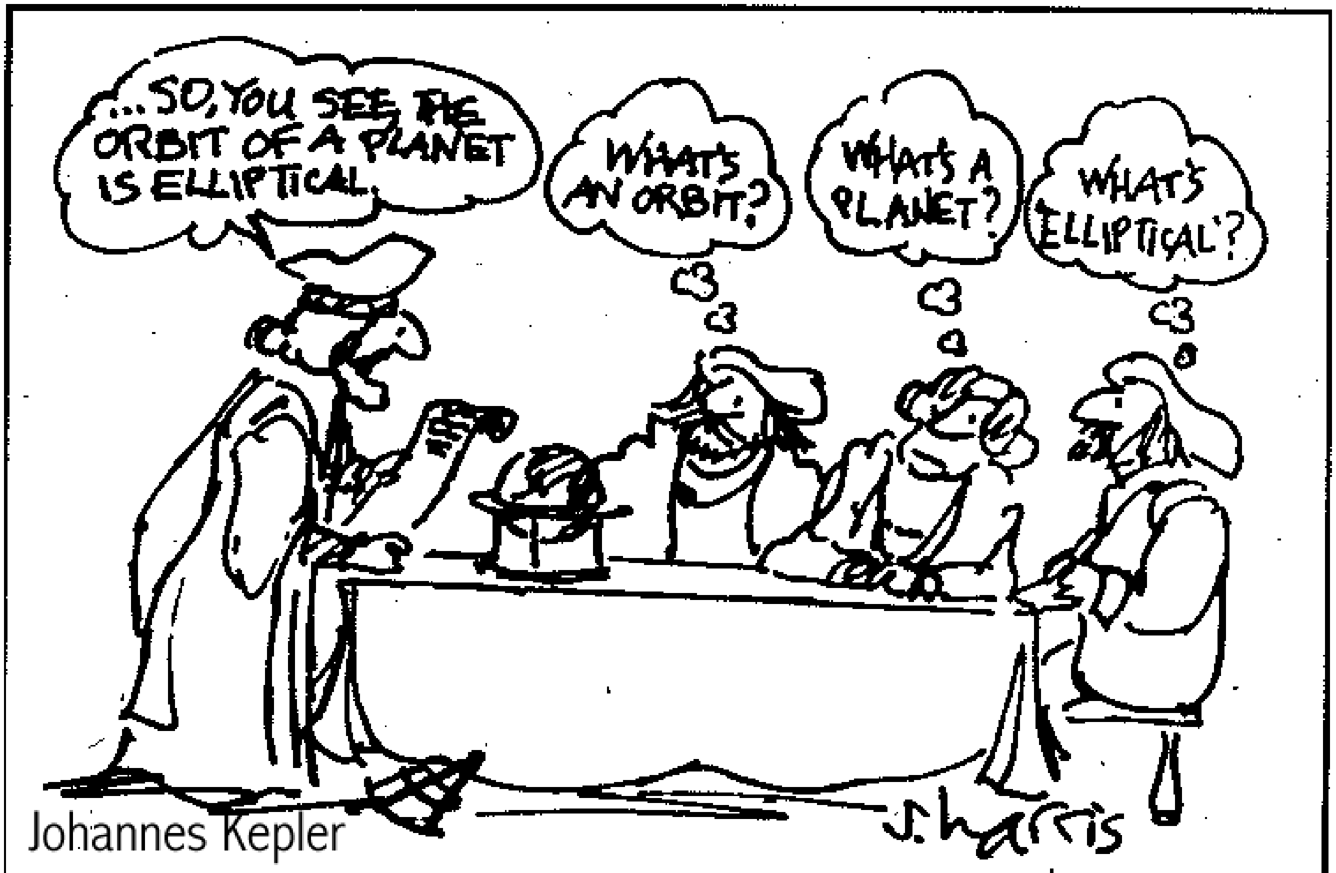




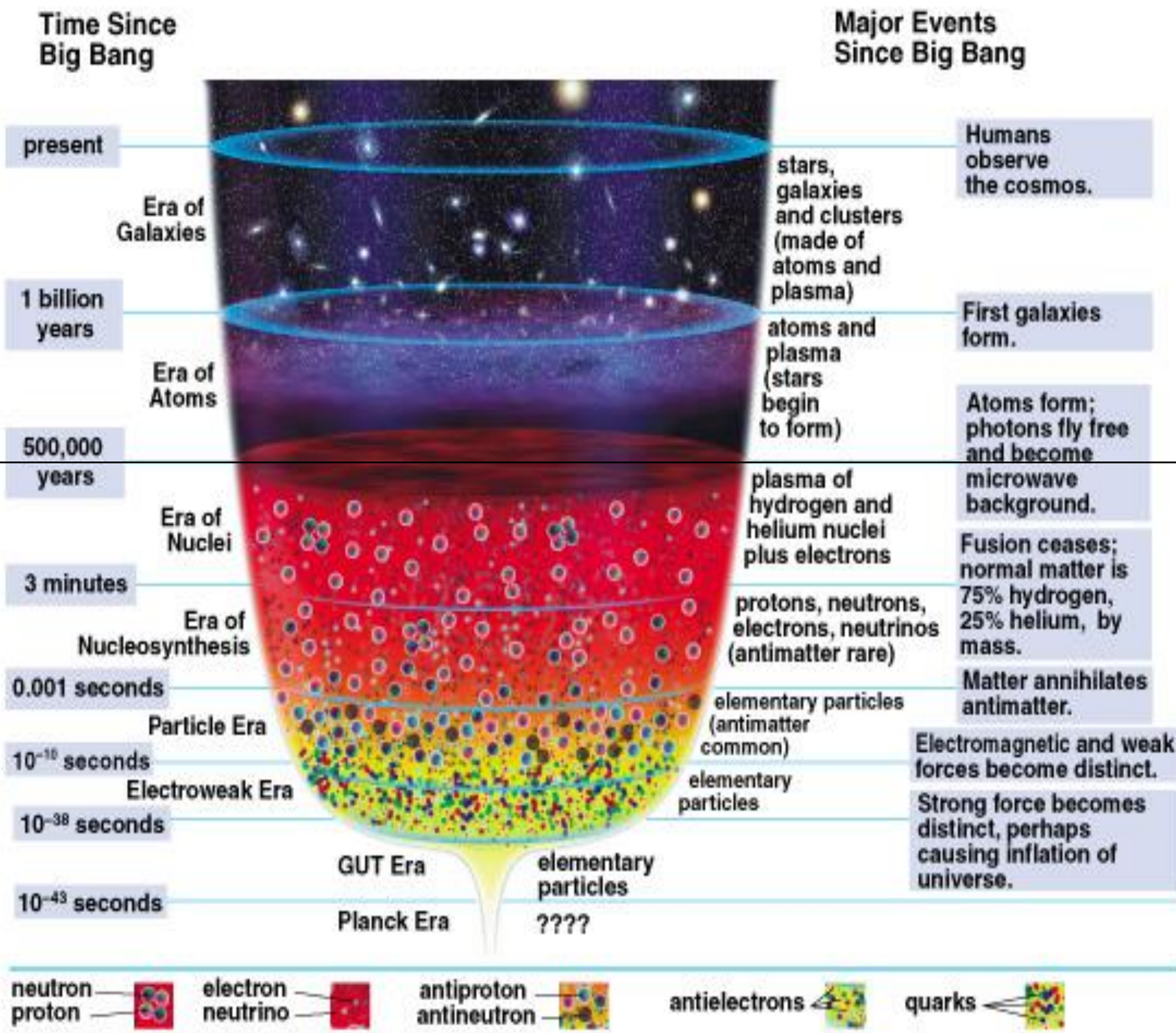
Putting it all together



Not Everyone Understands the Theory



History of the Universe



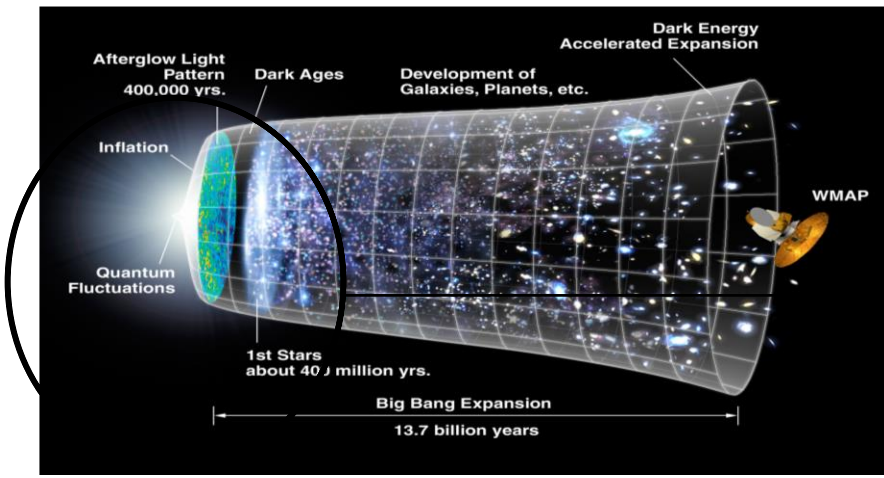
We have some idea, but don't know for sure how the universe is going to end yet.

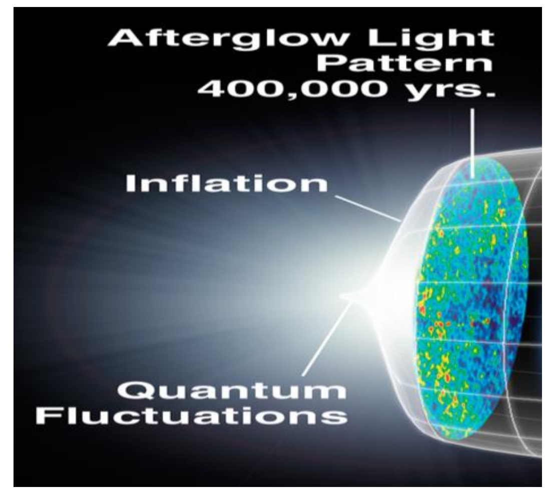
The observable universe

We know what's going on base on our knowledge of plasma physics and elementary particle physics

We still don't know how physics works in this era yet.

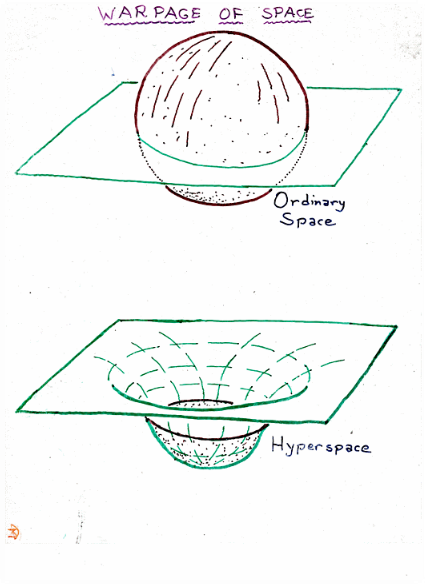
Inflation and Gravitational Waves

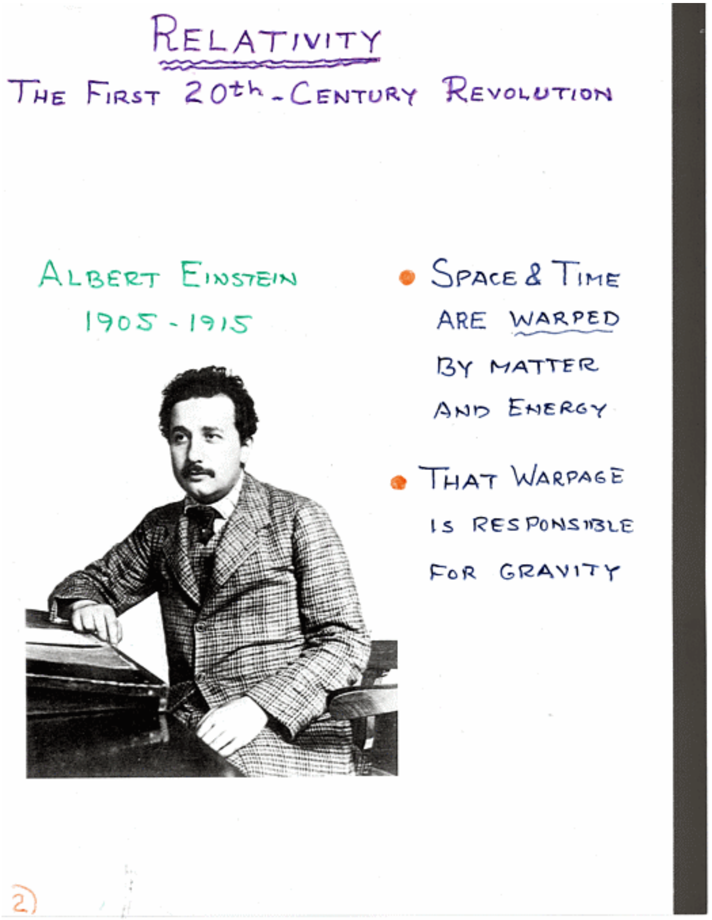




What are Gravitational Waves?

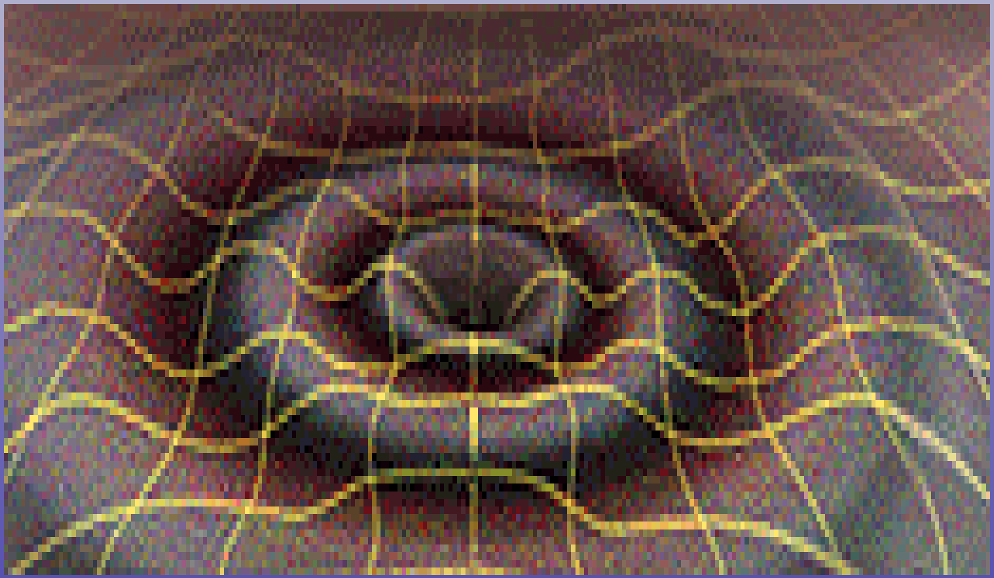
Gravitational Waves first appeared as part of Einstein's General Theory of Relativity





Einstein's Theory of General Relativity

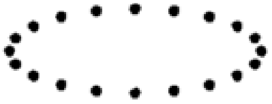
- Space-time tells matter how to move
- Matter tells space-time how to curve



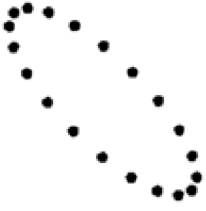
- **Gravitational Waves: Ripples in the fabric of space-time**
- **Black Holes: The final fate in the collapse of matter**

What Do Gravitational Waves Look Like?

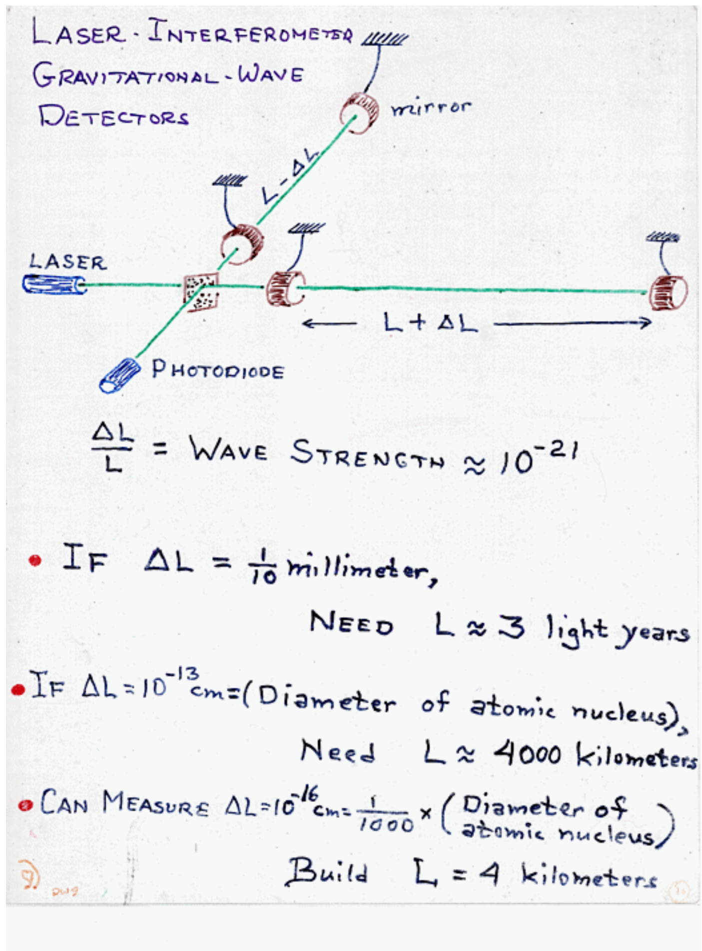
- Plus Polarization



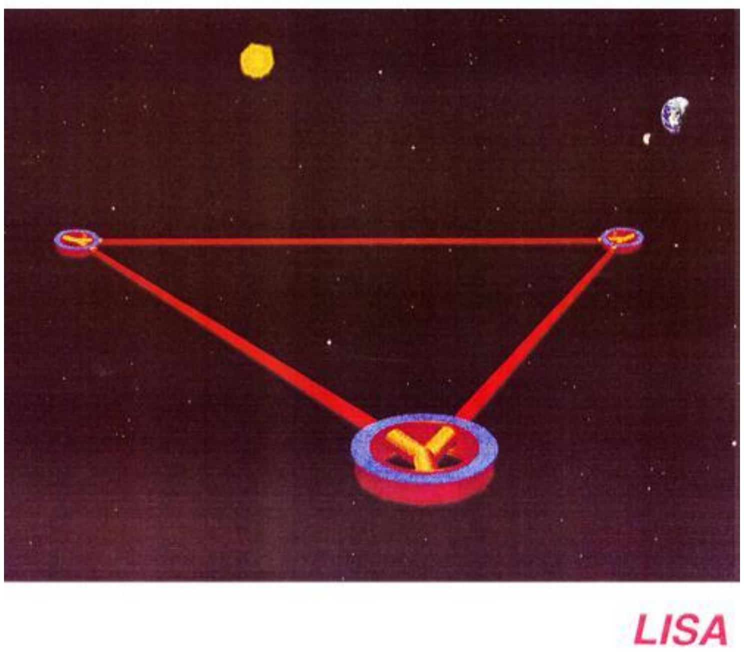
- Cross Polarization

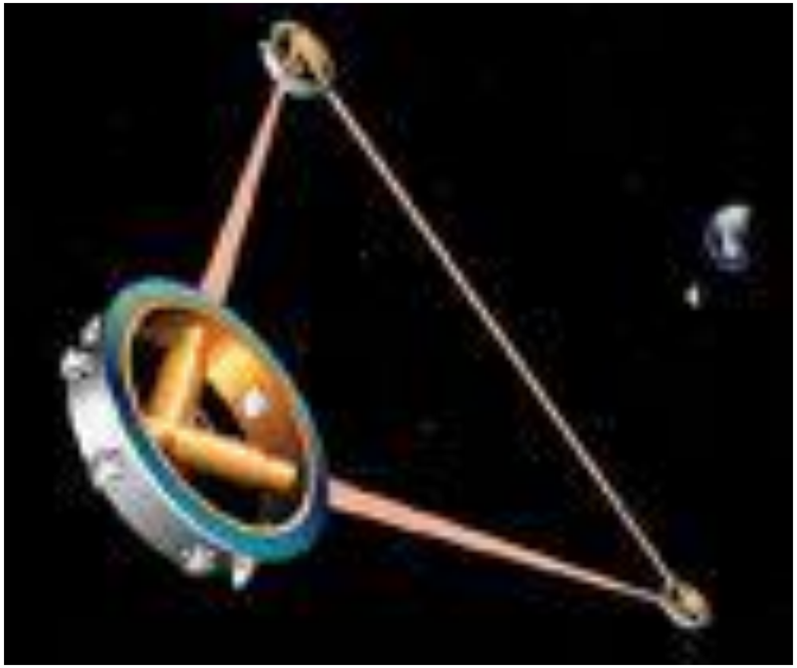


How GW interferometers work



LISA Space-based Gravitational Wave Observatory



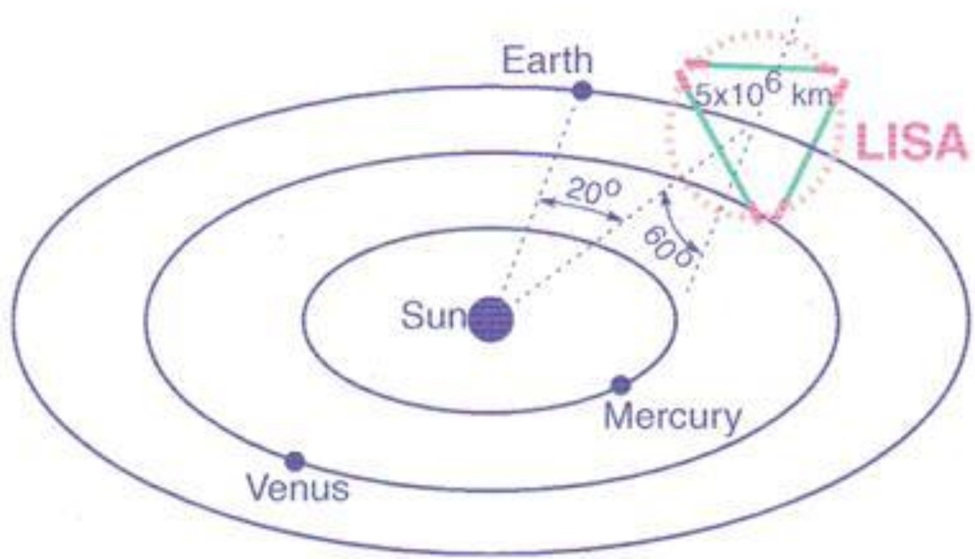


Low-Frequency Band: 0.1 to 0.0001 Hz

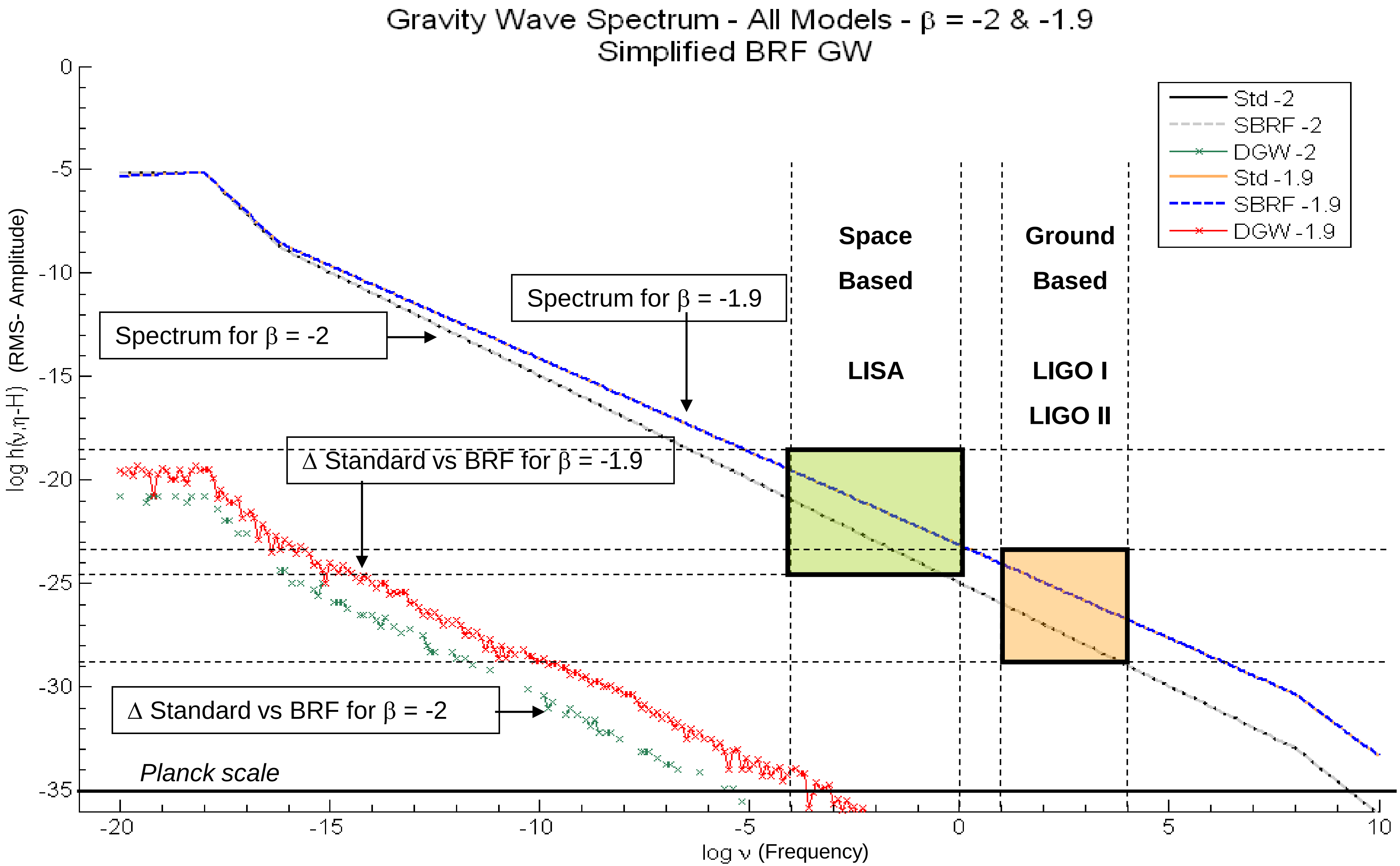
LISA

Laser Interferometer Space Antenna

L = 5 million kilometers



GW Spectrum
RMS Amplitude vs Frequency



Gravitational vs EM Radiation

GRAVITATIONAL WAVES CONTRASTED WITH ELECTROMAGNETIC WAVES

ELECTROMAGNETIC	GRAVITATIONAL
Oscillations of EM field propagating through spacetime	*Oscillations of the "fabric" of spacetime itself*
Incoherent superposition of waves from molecules, atoms, and particles	*Coherent emission by bulk motion of matter and energy*
Frequencies ~ 1 MHz and upward 20 orders	*Frequencies ~ 1 kHz and downward 20 orders*
Easily absorbed and scattered	*Never significantly absorbed or scattered*
Emitted from surfaces of objects (where optically thin and gravity is weak)	*Emitted most strongly by massive, compact, highly dynamical objects (where gravity is strong)*

IMPLICATIONS:

Gravitational waves are the ideal tool for probing strong-gravity regions of spacetime (general relativity)

Gravitational waves have the potential to bring us great surprises --- a "revolution" in our understanding of gravity and the Universe

Because of differences in EM and Gravitational Radiation, observing GWs is very different and so requires a different kind of astronomy

Why We Care about GWs

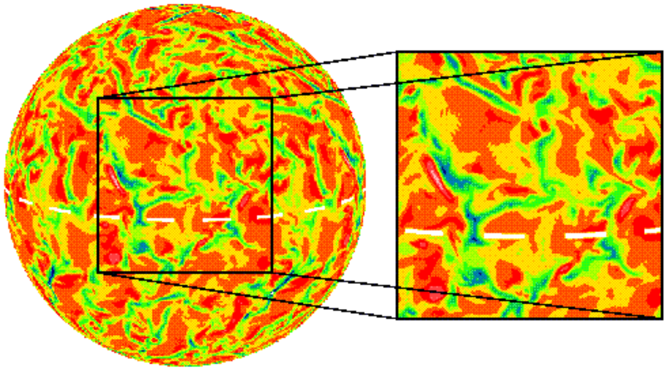
- Gravitational Waves can excite (turbulent?) modes of oscillation in the plasma field like a crystal is excited by sound waves.
- What are the results of these excited modes? What part did they play in the evolution of the universe?
- Can these excited modes contribute to the formation of structures in the early universe?

Magnetohydrodynamic (Plasma) Turbulence

- Plasma (ionized gas): charged-particles or magneto-fluid
- Plasma kinetic theory – particle description: Probability Density Function (p.d.f.) $f_j(\mathbf{x},\mathbf{p},t)$, $j = e^{-}$, *ions*.
- **M**agneto**H**ydro**D**ynamics (MHD) – $\mathbf{u}(\mathbf{x},t)$, $\mathbf{B}(\mathbf{x},t)$ and $p(\mathbf{x},t)$.
- MHD turbulence – **u**, **B** and *p* are random variables (mean & std. dev.).
- External magnetic fields & rotation affect plasma dynamics.

Homogeneous MHD Turbulence

- ☯ Examine flow in a small 3-D cube (3-torus).
- ☯ Assume periodicity and use Fourier series.
- ☯ *Homogeneous* means same statistics at different positions.
- ☯ Approximation that focuses on physics of turbulence.
- ☯ Periodic cube is a surrogate for a compact magneto-fluid.



Fourier Analysis

Represent velocity and magnetic fields in terms of Fourier coefficients;

$$\mathbf{u}(\mathbf{x},t) = \frac{1}{N^{3/2}} \sum_{\mathbf{k}} \tilde{\mathbf{u}}(\mathbf{k},t) \exp(i\mathbf{k}\cdot\mathbf{x}), \qquad \mathbf{k}\cdot\tilde{\mathbf{u}}(\mathbf{k},t) = 0$$

$$\mathbf{b}(\mathbf{x},t) = \frac{1}{N^{3/2}} \sum_{\mathbf{k}} \tilde{\mathbf{b}}(\mathbf{k},t) \exp(i\mathbf{k}\cdot\mathbf{x}), \qquad \mathbf{k}\cdot\tilde{\mathbf{b}}(\mathbf{k},t) = 0$$

Wave vector: $\mathbf{k} = (n_x, n_y, n_z)$, where $n_m \in \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Wave length: $\lambda_k = 2\pi/|\mathbf{k}|$. Numerically, we use only $0 < |\mathbf{k}| \leq K$.

In computational physics, this is called a ‘spectral method’.

Fourier-Transformed MHD Equations

Below, $\mathbf{Q}_u$ and $\mathbf{Q}_b$ are nonlinear terms involving products of the velocity and magnetic field coefficients. In "k-space", we have

$$\frac{d\tilde{\mathbf{u}}(\mathbf{k})}{dt} = \mathbf{Q}_u(\mathbf{k}) + 2\,\tilde{\mathbf{u}}(\mathbf{k}) \times \mathbf{\Omega} + i\mathbf{k} \cdot \mathbf{B}_o\,\tilde{\mathbf{b}}(\mathbf{k}) - \nu k^2 \tilde{\mathbf{u}}(\mathbf{k})$$

$$\frac{d\tilde{\mathbf{b}}(\mathbf{k})}{dt} = \mathbf{Q}_b(\mathbf{k}) + i\mathbf{k} \cdot \mathbf{B}_o\,\tilde{\mathbf{u}}(\mathbf{k}) - \eta k^2 \tilde{\mathbf{b}}(\mathbf{k}).$$

Direct numerical simulation (DNS) includes $\mathcal{N}$ modes with $\mathbf{k}$ such that $0 < |\mathbf{k}| \leq k_{max}$ and so defines a **dynamical system** of independent Fourier modes.

Non-linear Terms

The $\mathbf{Q}_u$ and $\mathbf{Q}_b$ are convolution sums in k-space:

$$\mathbf{Q}_u(\mathbf{k}) = \left(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{k}}\hat{\mathbf{k}}\right)\cdot \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \left[\tilde{\mathbf{u}}(\mathbf{p})\times\tilde{\boldsymbol{\omega}}(\mathbf{q}) + \tilde{\mathbf{j}}(\mathbf{p})\times\tilde{\mathbf{b}}(\mathbf{q})\right]$$

$$\mathbf{Q}_b(\mathbf{k}) = i\mathbf{k}\times \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \tilde{\mathbf{u}}(\mathbf{p})\times\tilde{\mathbf{b}}(\mathbf{q})$$

$$\tilde{\boldsymbol{\omega}}(\mathbf{q}) = i\mathbf{q}\times\tilde{\mathbf{u}}(\mathbf{q}), \qquad \tilde{\mathbf{j}}(\mathbf{p}) = i\mathbf{p}\times\tilde{\mathbf{b}}(\mathbf{p}).$$

Since $\nabla_{\mathbf{k}}\cdot\mathbf{Q}_u(\mathbf{k}) = \nabla_{\mathbf{k}}\cdot\mathbf{Q}_b(\mathbf{k}) = 0$, ideal MHD flows satisfy a Liouville theorem.

Statistical Mechanics of MHD Turbulence

- ‘Atoms’ are components of Fourier modes $\tilde{\mathbf{u}}(\mathbf{k})$, $\mathbf{b}(\mathbf{k})$.
- Canonical ensembles can be used (T.D. Lee, 1952).
- Gases have one invariant, the energy E.
- *Ideal* MHD ($\nu = \eta = 0$) has E, H_C and H_M.
- H_C and H_M are pseudoscalars under P or C or both.
- Ideal MHD statistics exists, but not same as $\nu, \eta \to 0+$.
- However, low-k ideal & real dynamics may be similar.

Ideal Invariants with Ω_o and B_o

3-D MHD Turbulence, with $\mathbf{\Omega}_o$ and $\mathbf{B}_o$ has various ideal invariants:

Case	Mean Field	Angular Velocity	Invariants
I	0	0	E, H_C, H_M
II	$\mathbf{B}_o \neq 0$	0	E, H_C
III	0	$\mathbf{\Omega}_o \neq 0$	E, H_M
IV	$\mathbf{B}_o \neq 0$	$\mathbf{\Omega}_o = \sigma \mathbf{B}_o$	E , H_P
V	$\mathbf{B}_o \neq 0$	$\mathbf{\Omega}_o \neq 0$ ($\mathbf{B}_o \times \mathbf{\Omega}_o \neq 0$)	E

In Case V, the ‘parallel helicity’ is $H_P = H_C - \sigma H_M$ ($\sigma = \Omega_o / B_o$).

Statistical Mechanics of Ideal MHD

$$E = \frac{1}{2N^3}\sum_{\mathbf{k}}\left[|\tilde{\mathbf{u}}(\mathbf{k})|^2 + |\tilde{\mathbf{b}}(\mathbf{k})|^2\right]$$

Ideal invariants:

$$H_C = \frac{1}{2N^3}\sum_{\mathbf{k}} \tilde{\mathbf{u}}(\mathbf{k})\cdot\tilde{\mathbf{b}}^*(\mathbf{k})$$

$$H_M = \frac{1}{2N^3}\sum_{\mathbf{k}} \frac{i}{k^2}\mathbf{k}\cdot\tilde{\mathbf{b}}(\mathbf{k})\times\tilde{\mathbf{b}}^*(\mathbf{k})$$

Phase Space Probability Density Function:

$$D = Z^{-1}\exp(-\alpha E - \beta H_C - \gamma H_M) = Z^{-1}\exp(-\Sigma_{\mathbf{k}}\, y^\dagger M y)$$

α, β, γ are `inverse temperatures'; $y^{\mathrm{T}} = (u_1,u_2,b_1,b_2)$

β, γ, H_C , H_M are pseudoscalars under P and C.

Eigenvariables

There is a unitary transformation in phase space such that

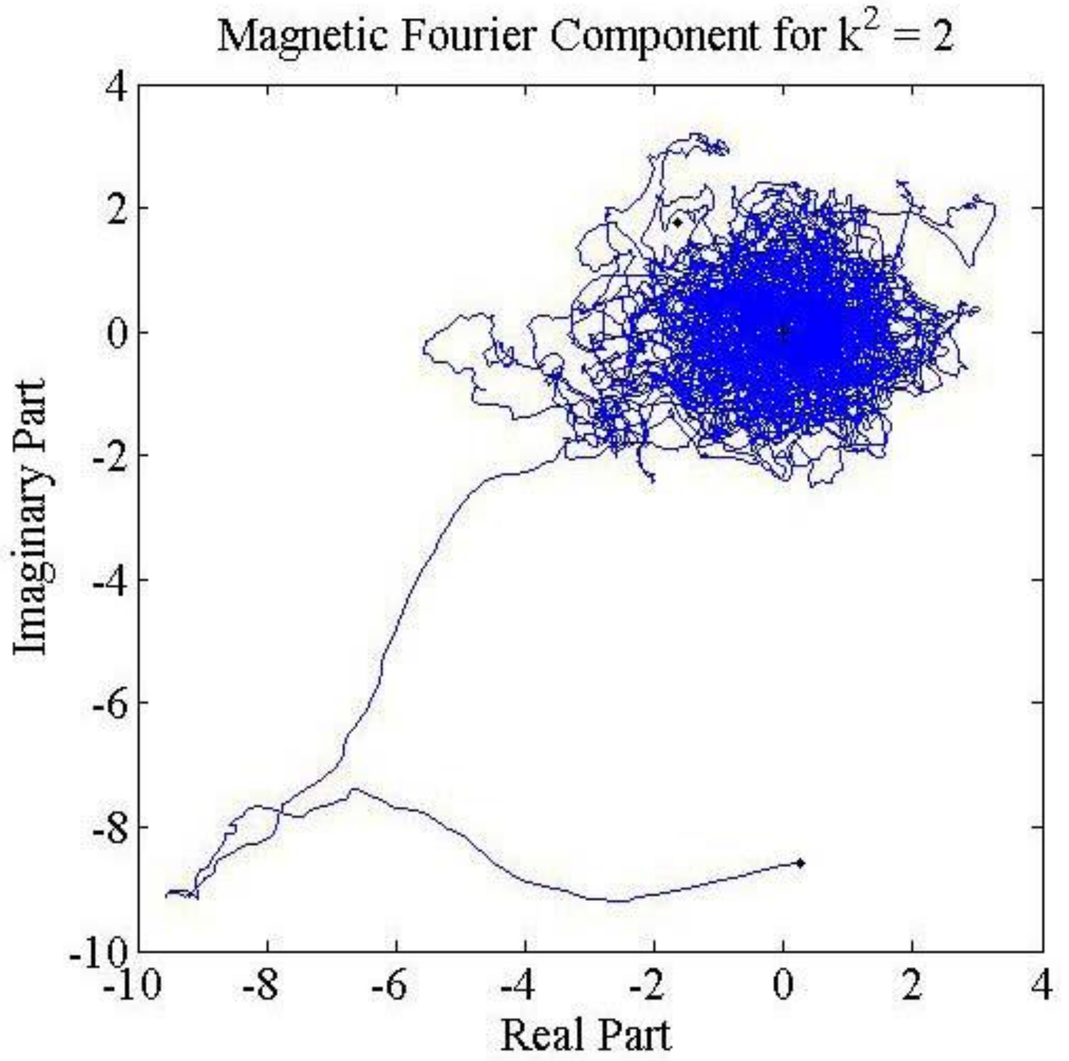
$$[u_1(\mathbf{k}), u_2(\mathbf{k}), b_1(\mathbf{k}), b_2(\mathbf{k})] \quad \rightarrow \quad [v_1(\mathbf{k}), v_2(\mathbf{k}), v_3(\mathbf{k}), v_4(\mathbf{k})]$$

$$D = \prod_{\mathbf{k}} D(\mathbf{k}) = \prod_{\mathbf{k}} \frac{1}{Z(\mathbf{k})} \exp\left(-\tfrac{1}{N^3} \sum_{j=1}^{4} \lambda_k^{(j)} \,|\, v_j(\mathbf{k}) \,|^2 \right)$$

The $v_j(\mathbf{k})$ are *eigenvariables* and the $\lambda_k^{(j)}$ are *eigenvalues* of the unitary transformation matrix.

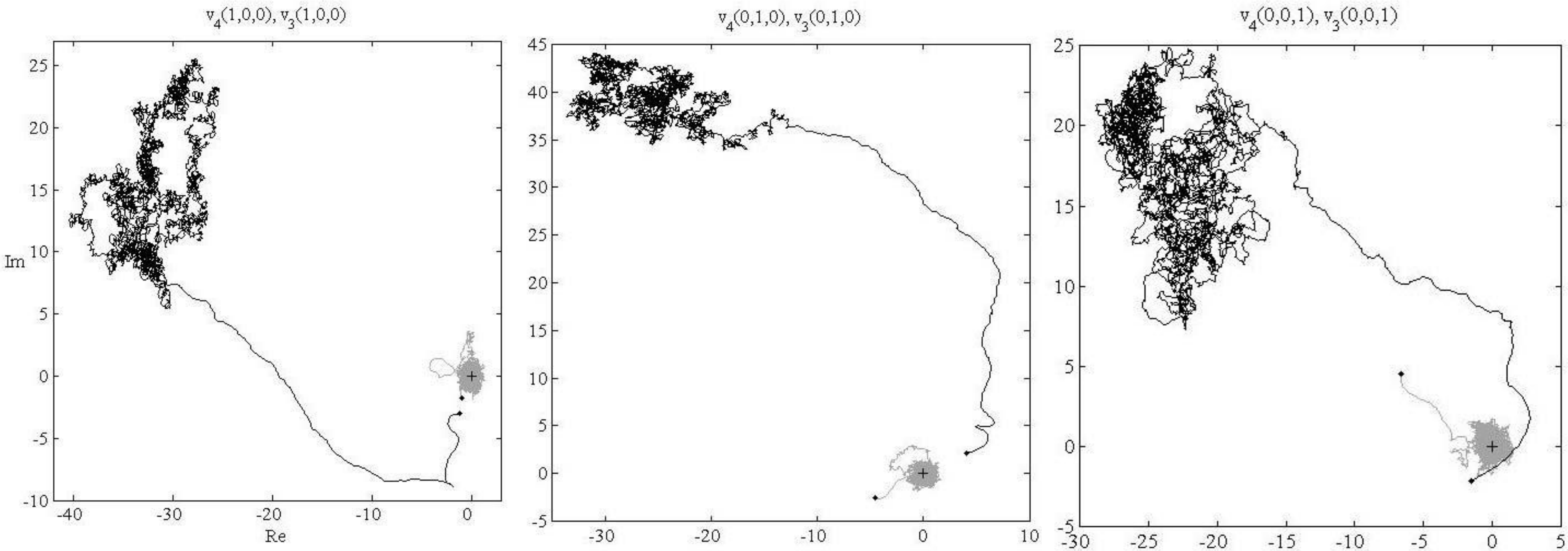
Phase Portraits

Although the dimension of phase space may be ~10^6, and the dynamics of the system is represented by a point moving on a trajectory in this space, we can project the trajectory onto 2-D planes to see it:



Coherent Structure, Case III (Rotating)

$$\alpha = 1.01862, \quad \beta = 0.00000, \quad \gamma = -1.017937$$



Non-ergodicity indicated by large mean values: time-averages ≠ ensemble averages.

Birkhoff-Khinchin Theorem: non-ergodicity = surface of constant energy disjoint.

Surface of constant energy is disjoint in ideal, homogeneous MHD turbulence.

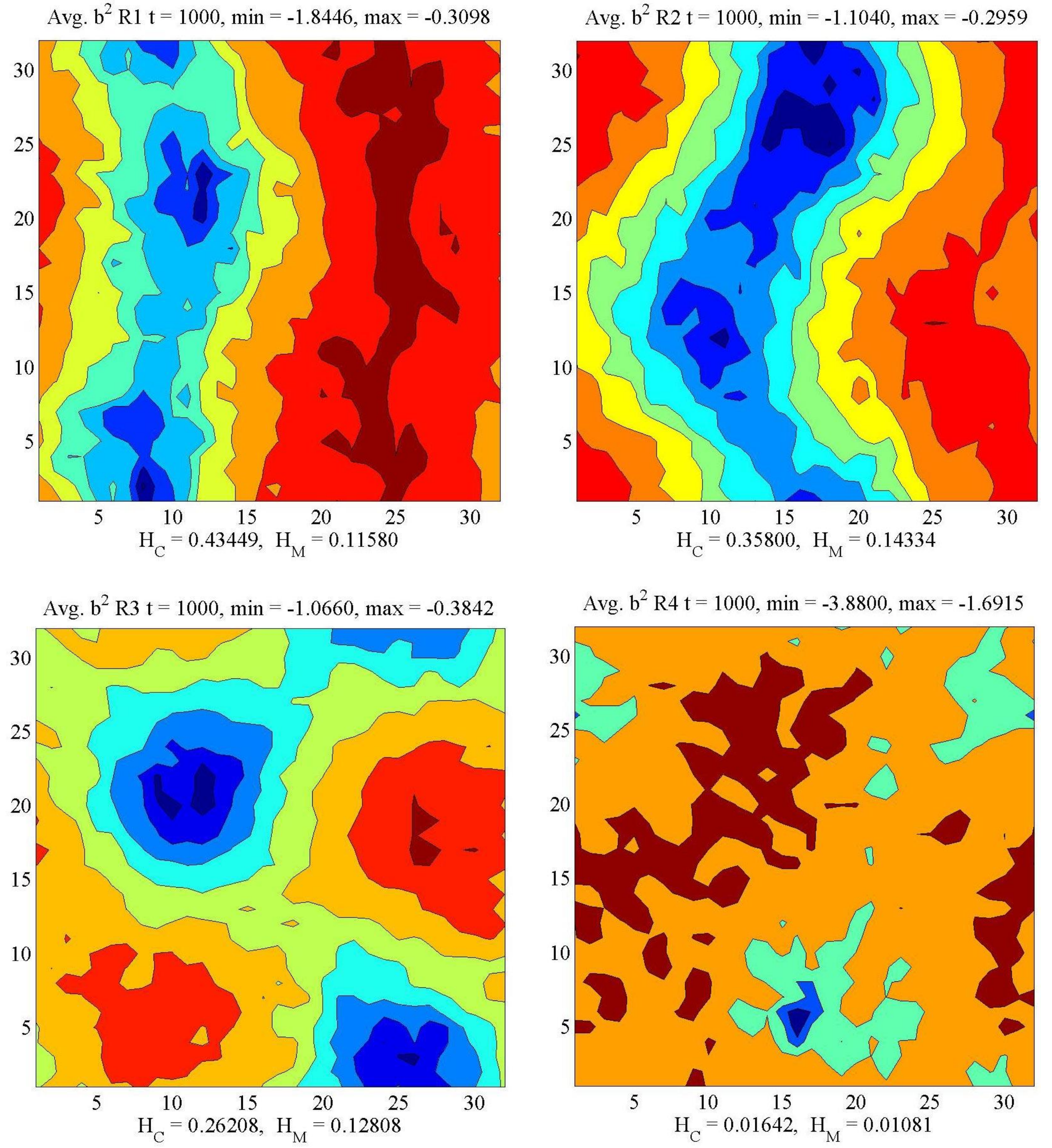
Coherent Structure in Physical Space

Case I Runs

$\Omega_o = B_o = 0$

Coherent **magnetic** energy density in the z = 15 plane of a 32^3 simulation

(averaged from t=0 to t=1000)



The Goal of This Work

- Apply the physics / mathematics of MHD Turbulence to Gravitational Waves / Relativistic Plasmas
- Demonstrate the formation of coherent structures (cosmic magnetic fields, density and temperature variations and relic gravitational waves) as a result of interactions with gravitational waves
- Utilize a GRMHD code to model both the plasma and the background space-time dynamically
- Study the interaction between MHD turbulence and gravitational waves and vice-versa

Our Approach

- Simulate the early universe after the inflationary event when the universe was populated by only a Homogeneous Plasma Field and Gravitational Radiation generated by inflation
- At this stage “classical” physics, General Relativity and Magneto-hydrodynamics, can describe the evolution of the universe
- We start with initial conditions at t = 3 min and evolve these conditions numerically using the GRMHD equations

GRMHD Variables - Spacetime

- Spacetime metric:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(\vec{x},t)(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

- Extrinsic Curvature:

$$K_{ij} = -\frac{1}{2\alpha}(\partial_t - \mathrm{L}_\beta)\gamma_{ij}(\vec{x},t)$$

- BSSN Evolution Variables:

$$\phi = \frac{1}{12}\ln[\det(\gamma_{ij})]$$

$$\tilde{\gamma}_{ij} = e^{-4\phi}\gamma_{ij}$$

$$K = \gamma^{ij} K_{ij}$$

$$\tilde{A}_{ij} = e^{-4\phi}(K_{ij} - \frac{1}{3}\gamma_{ij} K)$$

$$\tilde{\Gamma}^i = -\tilde{\gamma}^{ij}{}_{,j}$$

GRMHD Variables - MHD

$$\rho_* = \alpha\sqrt{\gamma}\rho_0 u^0 : conserved\ mass\ density$$

$$S_i = \alpha\sqrt{\gamma}T_i^0 : momentum\ density$$

$$\tau = \alpha^2\sqrt{\gamma}T^{00} - \rho_* : energy\ density$$

$$\tilde{B}^j = \sqrt{\gamma}B^j : magnetic\ field$$

$$v^i = \frac{1}{u^0}\gamma^{ij}u_j - \beta^i : 3-velocity$$

$$u^0 = \frac{1}{\alpha}\sqrt{1+\gamma^{ij}u_i u_j}$$

$$P = (\Gamma - 1)\rho_0\varepsilon : pressure$$

Stress-Energy Tensor

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \frac{8\pi G}{c^4} T^{\mu\nu} \; : Einstein's \;\; Eqn$$

$$T^{\mu\nu} = (\rho_0 h + b^2) u^\mu u^\nu + (P + \frac{b^2}{2}) g^{\mu\nu} - b^\mu b^\nu$$

$$h = 1 + \varepsilon + \frac{P}{\rho_0} \; : Enthalpy$$

$$b^\mu = \frac{1}{\sqrt{4\pi}} B^\mu_{(u)}$$

$$B^0_{(u)} = \frac{1}{\alpha} u_i B^i \quad ; \quad B^i_{(u)} = \frac{1}{u^0} (\frac{B^i}{\alpha} + B^0_{(u)} u^i)$$

Building our Model

- The observer is co-moving with fluid therefore $\alpha = 1$, $\beta = 0$, $u^i = (1,0,0,0)$
- Beginning of Classical Plasma Phase, t = 3 min
- $T = 10^9$ K, Plasma is composed of electrons, protons, neutrons, neutrinos and photons
- Mass-Energy density is 10^4 kg/m^3
- The universe is radiation-dominated
- The Hubble parameter at this time is 7.6×10^{16} km/s/Mpc

Other Parameters

- Age of the Universe 13.7 Billion Years
- Scale Factor: a(3.0 min) = 2.81×10^{-9}
- Specific Internal Energy, ε calculated from T
- Pressure, P: calculated using the Gamma Law with $\Gamma = 4/3$
- The Electric Field is set to zero b/c the observer is co-moving with the fluid
- The Magnetic field is set to 10^{-9} G based on theoretical estimates of the primordial seed field

Initial Spacetime

- Perturbed Robertson-Walker Metric

$$ds^2 = a(t)^2[-dt^2 + (\delta_{ij} + h_{ij})dx^i\, dx^j]$$

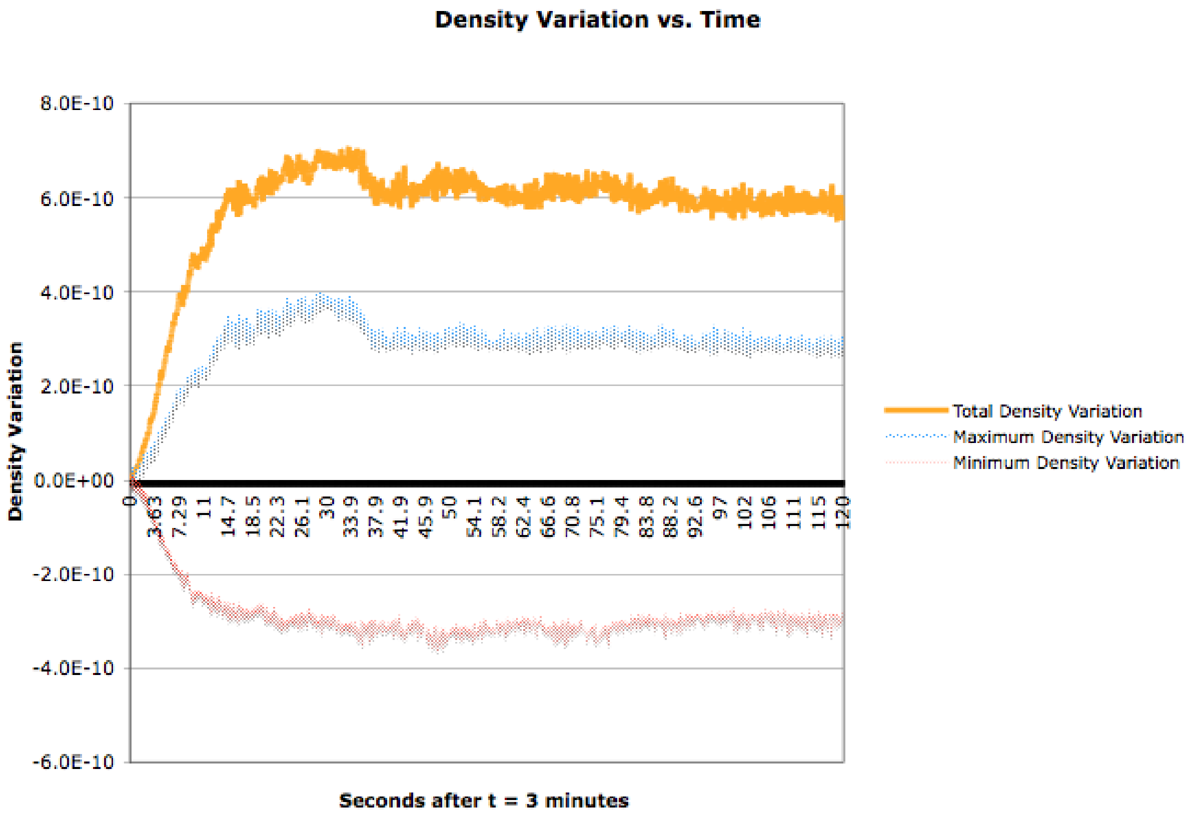
- Spectrum of Perturbations

$$h(k,t) = 8\sqrt{\pi}\, l_{pl}\, |1+\chi|^{-(1+\chi)} k^{2+\chi} / l_0$$

- Birefriengence

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu h_{ij}^L) = -2i\frac{\theta}{a}\dot{h}_{ij}^L \qquad \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu h_{ij}^R) = +2i\frac{\theta}{a}\dot{h}_{ij}^R$$

Preliminary Results



Future Developments

- Rewrite GR and GRMHD Equations in k-space so we can use spectral methods
- Add Viscosity
- Add Scalar Metric Perturbations
- Add Scalar Fields if needed
- Incorporate a Logarithmic Computational Grid

Questions?